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On the existence of resonances in the transmission probability for interactions arising from derivatives of Dirac's delta function

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Abstract

The scattering properties of regularizing finite-range potentials constructed in the form of squeezed rectangles, which approximate the first and second derivatives of the Dirac delta function $\delta(x)$, are studied in the zero-range limit. Particularly, for a *countable* set of interaction strength values, a *non-zero* transmission through the point potential $\delta'(x)$, defined as the weak limit (in the standard sense of distributions) of a special dipole-like sequence of rectangles, is shown to exist when the rectangles are squeezed to zero width. A tripole sequence of rectangles, which gives in the weak limit the distribution $\delta''(x)$, is demonstrated to exhibit the *total* transmission for a *countable* sequence of the rectangle's width that tends to zero. However, this tripole sequence does *not* admit a well-defined point interaction in the zero-range limit, making sense only for a *finite* range of the regularizing rectangular-like potentials.

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1. Introduction

Point or contact interactions are widely used in various applications to quantum physics (see [1, 2] and references therein including a large number of other publications, e.g., [3–8]). Intuitively, these interactions are understood as sharply localized potentials, exhibiting a number of interesting and intriguing features. Applications of these models to solid-state physics (see, e.g., some works by Exner and others [9–13]) are of particular interest nowadays, mainly because of the rapid progress in fabricating nanoscale quantum devices. Other applications arise in optics, for instance, in dielectric media where electromagnetic waves scatter at boundaries or thin layers [14].

The comprehensive and systematic presentation of the operator theory applied to the whole class of generalized point interactions has been given by Šeba [15], Gesztesy and Holden [16], and Albeverio *et al* [2]. According to this theory, the Hamiltonian for a general point interaction in the non-relativistic quantum mechanics is constructed as a self-adjoint extension of the corresponding free Hamiltonian with the interaction points removed, imposing a proper connection between the boundary conditions for the wavefunction and its first derivatives at the left and right sides of the singularity.

Thus, in the limit that neglects the interactions between electrons, the one-dimensional Schrödinger equation with a potential $V(x)$ for a stationary state reads

$$-\psi''(x) + V(x)\psi(x) = E\psi(x) \tag{1}$$

where the prime denotes the differentiation with respect to the spatial coordinate x, $\psi(x)$ is the wavefunction for a particle of mass m (we use units in which $\hbar^2/2m = 1$) and E is (positive, zero or negative) energy. The quantum mechanics of a single particle subject to the Dirac delta potential $V(x) = \delta(x)$ is a quite simple example in one dimension, while this interaction in two and three dimensions exhibits divergences. Similarly, the derivative of the Dirac delta function defined as a distribution

$$\delta'(x) := \mathrm{d}\delta(x)/\mathrm{d}x \tag{2}$$

e.g., on the space of test functions S, formed from all infinitely differentiable functions defined on the whole axis $-\infty < x < \infty$, which tend to zero for $|x| \to \infty$ as well as their derivatives of all orders, more rapidly than any power of $1/|x|$, causes a non-trivial problem for rigorous mathematical treatment even in one dimension. In the present paper, we study the scattering problem for this one-dimensional point dipole interaction, which we denote by

$$V_\sigma(x) = \sigma^2\delta'(x) \tag{3}$$

with $\sigma > 0$ being a dimensionless parameter. For reasons of notation in this paper we represent the interaction strength constant as a squared coefficient; clearly, it is sufficient to consider only the sign '+' in front of σ^2.

In order to avoid any confusion in this paper, it should be emphasized that in the literature the so-called δ' (or point dipole) interaction is used by different authors as different objects. Thus, in the pioneering work by Albeverio *et al* [2], Šeba [15] and Gesztesy and Holden [16] the δ' interaction has been defined assuming a discontinuity of wavefunctions but their continuous derivatives at the singularity point ($x = 0$). However, these connection conditions, which are ascribed to the δ' interaction, are in general incompatible with the standard definition of the interaction (3) through the $\delta'(x)$ function (2) being an odd function of x. Therefore, as already emphasized in the literature, e.g., by Exner [11] and Coutinho *et al* [23], the δ' interaction defined through discontinuous wavefunctions but their continuous derivatives have little resemblance to what the function (2) means. Our study presented here deals *only* with the definition (2) and therefore in our case of the δ' interaction we shall always furnish it with the argument x as in (2) or (3).

It should also be mentioned that using different scaling limits, the point dipole interaction with the boundary conditions of Albeverio *et al* [2] (discontinuity of wavefunctions but continuity of their derivatives) has been analysed in a mathematically proper way by Šeba [15]. The main point of his studies is that one has to introduce properly a renormalization of the coupling constant to deal with a real-point interaction but not with two independent systems. To this end, Šeba has defined the renormalized δ' interaction as an $\epsilon \to 0$ limit of the following family of regularized potentials:

$$V_{\nu,\sigma,\epsilon}(x) = (\sigma^2/2\epsilon^\nu)[\delta(x-\epsilon) - \delta(x+\epsilon)] \qquad \nu > 0 \quad \epsilon > 0. \tag{4}$$

As a result, Šeba has proved that in the limit $\epsilon \to 0$ the interaction disappears if $\nu < 1/2$, appears as a $\delta(x)$ potential for $\nu = 1/2$ and becomes a totally reflecting wall at $x = 0$ if $\nu > 1/2$ splitting the system into two independent subsystems lying on the half-lines $(-\infty, 0)$ and $(0, \infty)$. Later the δ' interaction attracted considerable attention in both the physically and mathematically oriented literature resulting in important contributions to the theory of point interactions [17–28]. Particularly, in these studies the δ' interaction was considered as an example of a general point interaction parametrized by four real variables that connect the boundary conditions for the wavefunction at the left and right sides of the point interaction. However, a particular example, or more precisely, a special zero-range limit, which we study in detail in the present paper, was not considered there, but it is of relevance to the recent studies by Albeverio *et al* [29] and Tsutsui *et al* [30, 31] where the $\delta'(x)$ interactions (2) and (3) can be considered as a particular example fulfilling the conditions of their theorems.

In this paper, instead of using the approximation by two Dirac delta functions centred symmetrically with respect to the origin $x = 0$ like (4), we regularize the $\delta'(x)$ function by the functions which have a *non-zero* discontinuity at $x = 0$. At this level of a 'small but finite' regularizing parameter ϵ, one can get a discontinuity of both the wavefunction and its derivative developing around $x = 0$ in the limit $\epsilon \to 0$. Despite the unrenormalized case $(\nu = 1)$ being shown not interesting from the physical point of view (because as proved by Šeba [15] and calculated later by Patil [32] the δ' interaction was shown to act in fact as a non-transparent barrier), nevertheless we focus on this case in the present paper using a very particular sequence of regularizing potentials.

To motivate our point of view that the transparency properties of the point dipole interaction (3) should be revised, let us consider two types of regularizing families $\{V_{\sigma,\epsilon}(x)\}$, which could possibly result in different behaviour of the wavefunction as $\epsilon \to 0$ if we impose different constraints on the behaviour of each function $V_{\sigma,\epsilon}(x)$ in the vicinity of $x = 0$. For the first type, this function is supposed to be identically zero in the vicinity of $x = 0$, choosing, e.g., the double-delta approximation (4) or appropriate step functions, the support of which *does not contain* the point $x = 0$. Then for each $\epsilon > 0$, equation (1) has an oscillatory solution at both the left and right sides of $x = 0$, which as can easily be shown explicitly [23, 32] collapses to a continuous function with a node at $x = 0$ as $\epsilon \to 0$. As a result an incident current is *totally* reflected at this point, even though for each finite ϵ the energy flow is partially transmitted. For the second type of regularizing functions let us impose an opposite constraint at $x = 0$, namely the existence of a *finite* discontinuity at this point that goes to infinity as $\epsilon \to 0$. As an example, the step function $\Delta'_\epsilon(x)$ defined by

$$\Delta'_\epsilon(x) := \begin{cases} 0 & \text{if} \quad -\infty < x < -\epsilon \\ \epsilon^{-2} & \text{if} \quad -\epsilon < x < 0 \\ -\epsilon^{-2} & \text{if} \quad 0 < x < \epsilon \\ 0 & \text{if} \quad \epsilon < x < \infty \end{cases} \tag{5}$$

and shown in figure 1, which splits the x axis into four regions and has a finite dipole-like form, can be adopted. With this potential, two different types of solution are connected at the origin $x = 0$, namely tunnelling and oscillatory ones. For each finite ϵ the wavefunction $\psi_\epsilon(x)$ as a solution to the Schrödinger equation (1) with $E > 0$ and the potential (5) has no node at $x = 0$, whereas the discontinuity jump of the regularizing sequence increases very sharply (as ϵ^{-2}) with $\epsilon \to 0$. Therefore, one could expect that at least for some values of σ the limit of the continuous function $\psi_\epsilon(x)$ will have a *finite* discontinuity at this point as $\epsilon \to 0$. This would immediately imply a non-zero current flow across the $\delta'(x)$ potential as we show in the present paper at least for a particular rectangular-like sequence.

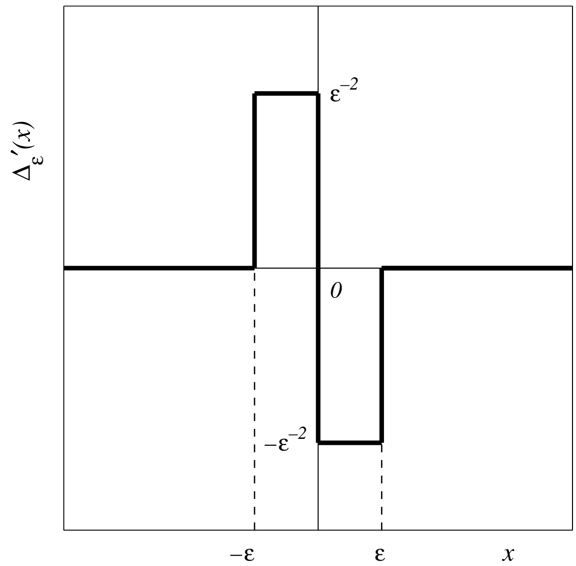

Figure 1. The rectangular model (5) of the $\delta'(x)$ function with ϵ being a regularizing parameter.

Both ways of obtaining the $\delta'(x)$ distribution defined as the $\epsilon \to 0$ limits of the double-delta family (4) with $\nu = 1$ and of the rectangular sequence (5) can be arranged through a repeated limit of a reconstructed rectangular sequence if we introduce additionally some distance l between two rectangles of opposite 'polarity' as a second regularizing parameter. For instance, one can choose the following stepwise function:

$$\Delta'_{\epsilon,l}(x) = \begin{cases} \pm(\epsilon l)^{-1} & \text{if} \quad -(\epsilon \pm l)/2 < x < (\epsilon \mp l)/2 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

constructed from rectangles as shown in figure 2, which contain two regularizing parameters ϵ and l. Then the $\delta'(x)$ function (2) can be represented as repeated limits in the weak topology (on the space S), either

$$\delta'(x) = \lim_{\epsilon\to 0}\lim_{l\to\epsilon} \Delta'_{\epsilon,l}(x) \tag{7}$$

or

$$\delta'(x) = \lim_{l\to 0}\lim_{\epsilon\to 0} \Delta'_{\epsilon,l}(x). \tag{8}$$

In other words, the scattering problem for the $\delta'(x)$ interaction can be solved by using two different limiting procedures, namely (7) or (8). The key point of our study is the use of the limit (7) instead of the limit (8). The striking feature is that these ways give different reflection and transmission properties of the $\delta'(x)$ interaction obtained in each of these limits.

Except for the $\delta'(x)$ interaction (2), here we also study the transmission problem for the $\delta''(x)$ interaction defined similarly as the second derivative of the Dirac delta function, i.e.,

$$\delta''(x) := \mathrm{d}^2\delta(x)/\mathrm{d}x^2. \tag{9}$$

Since for the $\delta'(x)$ interaction (2) we use the rectangular sequence (5) which results in a non-trivial physics, it would be of interest to use a similar but tripole sequence of rectangles that gives in the weak limit (on the space S) the $\delta''(x)$ distribution. Therefore for the interaction

$$V_g(x) = g\delta''(x) \tag{10}$$

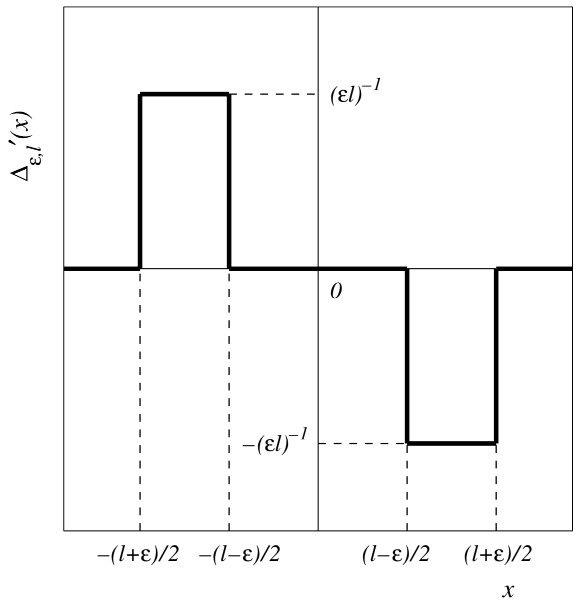

Figure 2. The rectangular model (6) of the $\delta'(x)$ function with two regularizing parameters ϵ and l.

we choose the following stepwise function:

$$\Delta''_\epsilon(x) := \begin{cases} 0 & \text{if} \quad -\infty < x < -3\epsilon/2 \\ \epsilon^{-3} & \text{if} \quad -3\epsilon/2 < x < -\epsilon/2 \\ -2\epsilon^{-3} & \text{if} \quad -\epsilon/2 < x < \epsilon/2 \\ \epsilon^{-3} & \text{if} \quad \epsilon/2 < x < 3\epsilon/2 \\ 0 & \text{if} \quad 3\epsilon/2 < x < \infty \end{cases} \tag{11}$$

as shown in figure 3.

Then in the same sense of distributions we have the limit

$$\delta''(x) = \lim_{\epsilon \to 0} \Delta''_\epsilon(x). \tag{12}$$

It is worth mentioning here that a *tripole* approximation that uses three Dirac delta functions separated by small distances ϵ has already been applied by Cheon and Shigehara [24] for the construction of a family of point interactions at one point ($x = 0$) as $\epsilon \to 0$. They started from the following triple-delta function:

$$\xi(x; u, v, \epsilon) = v(\epsilon)\delta(x + \epsilon) + u(\epsilon)\delta(x) + v(\epsilon)\delta(x - \epsilon) \tag{13}$$

where the behaviour of $u(\epsilon)$ and $v(\epsilon)$ as functions of ϵ was chosen in a proper way to get a discontinuity of wavefunctions as $\epsilon \to 0$ and as a result to obtain a well-defined non-trivial point interaction with each choice of these functions. In the particular case when $v = -u/2 = \epsilon^{-2}$ the triple-delta function (13) also tends in the weak topology to the $\delta''(x)$ distribution defined by (9). However, the calculation of the scattering amplitudes for the $\delta''(x)$ potential (10) carried out by Patil [32] leads to the trivial result, i.e., to complete separation of the whole system by this interaction into two subsystems. On the other hand, the tripole but rectangular-like approximation (11) gives interesting physics; however, only for $\epsilon > 0$ because there is no limit for the scattering amplitudes of the sequence (11) as $\epsilon \to 0$.

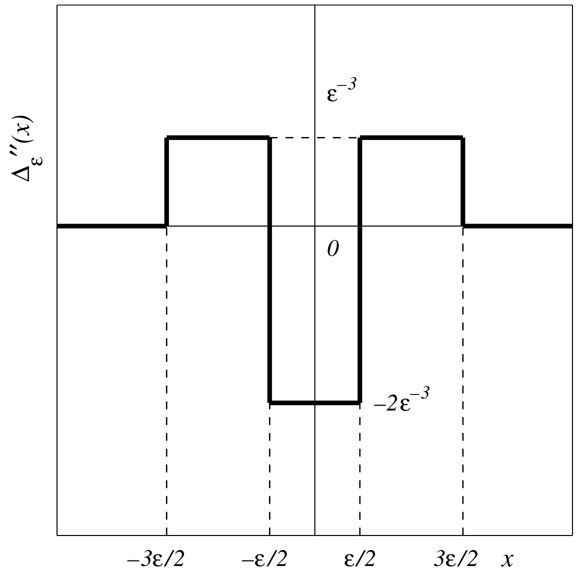

Figure 3. The rectangular model (11) of the $\delta''(x)$ function with ϵ being a regularizing parameter.

2. Non-zero transmission probability of the $\delta'(x)$ potential

The Schrödinger equation (1) with the rectangular approximation (5) can explicitly be solved in all four regions where the function $\Delta''_\epsilon(x)$ is constant. Using then the continuity of the wavefunction $\psi_\epsilon(x)$ and its derivative $\psi'_\epsilon(x)$ at the boundaries $x = 0$ and $x = \pm\epsilon$, one finds the following positive-energy solution:

$$\psi_\epsilon(x) = \begin{cases} e^{ikx} + R\,e^{-ikx} & \text{if} \quad -\infty < x < -\epsilon \\ 2D^{-1}\,e^{-i\eta}\big[\big(\cos\alpha - i\frac{\eta}{\alpha}\sin\alpha\big)\cosh\frac{\beta x}{\epsilon} & \\ \quad + \frac{\alpha}{\beta}\big(\sin\alpha + i\frac{\eta}{\alpha}\cos\alpha\big)\sinh\frac{\beta x}{\epsilon}\big] & \text{if} \quad -\epsilon < x < 0 \\ 2D^{-1}\big\{\cos\big[\alpha\big(\frac{x}{\epsilon} - 1\big)\big] + i\frac{\eta}{\alpha}\sin\big[\alpha\big(\frac{x}{\epsilon} - 1\big)\big]\big\} & \text{if} \quad 0 < x < \epsilon \\ T\,e^{ikx} & \text{if} \quad \epsilon < x < \infty \end{cases} \tag{14}$$

where $k = \sqrt{E}$, $\eta = \epsilon k$ and

$$D = 2\cos\alpha\cosh\beta + \left(\frac{\beta}{\alpha} - \frac{\alpha}{\beta}\right)\sin\alpha\sinh\beta - i\left(\frac{\alpha}{\eta} + \frac{\eta}{\alpha}\right)\sin\alpha\cosh\beta + i\left(\frac{\beta}{\eta} - \frac{\eta}{\beta}\right)\cos\alpha\sinh\beta \tag{15}$$

with

$$\alpha = \sqrt{\sigma^2 + \eta^2} \qquad \text{and} \qquad \beta = \sqrt{\sigma^2 - \eta^2}. \tag{16}$$

Here the reflection (from the left) coefficient R and the transmission coefficient T are given by

$$R = \frac{e^{-2i\eta}}{D}\left[-\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right)\sin\alpha\sinh\beta + i\left(\frac{\alpha}{\eta} - \frac{\eta}{\alpha}\right)\sin\alpha\cosh\beta - i\left(\frac{\eta}{\beta} + \frac{\beta}{\eta}\right)\cos\alpha\sinh\beta\right] \tag{17}$$

$$T = \frac{2}{D}\,\mathrm{e}^{-2\mathrm{i}\eta}. \tag{18}$$

Note that the energy dependence in all the quantities (15)–(18) appears only through the dimensionless parameter η. In straightforward way one can check that the equality $|R|^2 + |T|^2 = 1$ holds.

Once the solution with finite ϵ is found, we simply let ϵ tend towards zero. As a result, in the limit $\eta \to 0$ that corresponds to the limit $\epsilon \to 0$ we find that $\alpha, \beta \to \sigma$ and $D \to \infty$, so that

$$R \to -1 \qquad \text{and} \qquad T \to 0 \tag{19}$$

except for those values of σ for which

$$\lim_{\eta\to 0} \eta^{-1}(\alpha \sin\alpha \cosh\beta - \beta\cos\alpha \sinh\beta) = 0. \tag{20}$$

These values form a countable set $\{\sigma_n\}_{n=0}^{\infty}$ being the roots of the equation

$$\tan\sigma = \tanh\sigma. \tag{21}$$

In physical terms, equation (21) except for the trivial solution $\sigma_0 = 0$ admits the series of discrete values σ_n, $n = 1, 2, \ldots$, at which the $\delta'(x)$ barrier becomes partially transparent. Note that the solution obtained from the second limit (8) does not give these σ_n. Using next the limit (20) and equation (21), for each σ_n, $n = 0, 1, \ldots$, one obtains the limit $D \to 2\cos\sigma_n \cosh\sigma_n$, so that for each strength level σ_n we have the following limiting values for the reflection and transmission coefficients:

$$R = -\tanh^2\sigma_n \qquad \text{and} \qquad T = \sec\sigma_n\,\mathrm{sech}\,\sigma_n. \tag{22}$$

Similarly, at these σ_n the energy that penetrates through the $\delta'(x)$ potential is $|T|^2 = 1 - \tanh^4\sigma_n$.

The boundary values for the wavefunction and its derivative can also be calculated explicitly for all $\sigma = \sigma_n$, $n = 1, 2, \ldots$. These values are easily calculated explicitly from the solution given by (14)–(18) when the limits from the barrier and well sides of the singularity are implemented as follows:

$$\psi_B := \psi(-0) = \lim_{\epsilon\to 0} \psi_\epsilon(-\epsilon) = \mathrm{sech}^2\sigma_n \tag{23}$$

$$\psi_W := \psi(+0) = \lim_{\epsilon\to 0} \psi_\epsilon(\epsilon) = \sec\sigma_n\,\mathrm{sech}\,\sigma_n \tag{24}$$

$$\psi'_B := \psi'(-0) = \lim_{\epsilon\to 0} \psi'_\epsilon(-\epsilon) = \mathrm{i}k\sec^2\sigma_n \tag{25}$$

$$\psi'_W := \psi'(+0) = \lim_{\epsilon\to 0} \psi'_\epsilon(\epsilon) = \mathrm{i}k\sec\sigma_n\,\mathrm{sech}\,\sigma_n. \tag{26}$$

Thus, at the point $x = 0$ for each $\sigma = \sigma_n$ the jump of discontinuity of the wavefunction $\psi(x)$ becomes

$$\Delta\psi := \psi_W - \psi_B = \mathrm{sech}\,\sigma_n(\sec\sigma_n - \mathrm{sech}\,\sigma_n) \tag{27}$$

whereas the jump of discontinuity of the derivative $\psi'(x)$ is

$$\Delta\psi' := \psi'_W - \psi'_B = \mathrm{i}k\sec\sigma_n(\mathrm{sech}\,\sigma_n - \sec\sigma_n). \tag{28}$$

According to the Albeverio–Dąbrowski–Kurasov (ADK) theorem [29], the family of boundary conditions connected through the matrix equation

$$\begin{pmatrix}\psi(+0)\\ \psi'(+0)\end{pmatrix} = \mathrm{e}^{\mathrm{i}\theta}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\psi(-0)\\ \psi'(-0)\end{pmatrix} \tag{29}$$

with the real parameters $\theta \in [0, \pi)$, $a, b, c, d \in \mathbf{R}$ fulfilling the condition $ad - bc = 1$ (called non-separated boundary conditions), correspond to the set of self-adjoint extensions of the operator $-\mathrm{d}^2/\mathrm{d}x^2$ defined in the Hilbert space $L_2(\mathbf{R}\backslash\{0\})$. Using the boundary conditions (23)–(26), one can easily find the matrix coefficients:

$$\theta = 0 \qquad a = \frac{1}{d} = \frac{\cosh \sigma_n}{\cos \sigma_n}. \tag{30}$$

In this regard it is important to note here that the countable set of self-adjoint extensions for the potential $\delta'(x)$ determined by the family (30) is *not* identical to that of the potential $\delta^{(1)}(x)$ [29] which admits another parametrization. However, this is not surprising due to the fact that the distribution $\delta^{(1)}(x)$ belongs to the space K' defined on the space of test functions with a jump discontinuity at the origin K [33], while the $\delta'(x)$ function (2) makes sense as a distribution from the space S' $(\ldots S \subset K \subset L_2 \subset K' \subset S' \ldots)$. In the context of the ADK theorem Šeba's result can also be considered as another particular self-adjoint extension with separated boundary conditions [29]. In other words, the boundary conditions at $x = 0$ for the point interaction (3) appear to be non-separated only for those σ which fulfil equation (21).

Instead of the relation (29), another connection between the boundary conditions has been introduced by Tsutsui, Fülöp and Cheon (TFC) in the form of the following matrix equation [30, 31]:

$$(U - I)\begin{pmatrix} \psi(+0) \\ \psi(-0) \end{pmatrix} + \mathrm{i}L_0(U + I)\begin{pmatrix} \psi'(+0) \\ -\psi'(-0) \end{pmatrix} = 0 \tag{31}$$

with some constant $L_0 \neq 0$ of length dimension, where $U \in \mathrm{U}(2)$ and I denotes the unit matrix in U(2). Within the TFC family, the matrix U for our case reads

$$U = \begin{pmatrix} -R & T \\ T & R \end{pmatrix} \tag{32}$$

where the reflection and transmission coefficients R and T are given by (22). Equation (32) is fulfilled for any constant L_0 if the boundary conditions for the wavefunction are given by the limits (23)–(26).

Finally, we note that the connection between the boundary conditions can also be rewritten in the penetrable case ($\sigma = \sigma_n, n = 1, 2, \ldots$) in the form

$$\psi_B \psi_B' = \psi_W \psi_W' \tag{33}$$

or through the ratios

$$\frac{\psi_B}{\psi_W} = \frac{\psi_W'}{\psi_B'} = \frac{\cos \sigma_n}{\cosh \sigma_n} < 1. \tag{34}$$

The last inequality means that the probability of finding an electron near $x = 0$ is higher from the well side as intuitively expected.

On the other hand, for all $\sigma \neq \sigma_n$ owing to (19) one immediately obtains the separated boundary conditions: $\psi_B = \psi_W = 0$, $\psi_B' = 2\mathrm{i}k$ and $\psi_W' = 0$. Thus, for arbitrary $\sigma \neq \sigma_n$ the wavefunction $\psi(x)$ is continuous at $x = 0$, whereas its derivative $\psi'(x)$ is discontinuous as discussed in the previous studies [32]. However, for $\sigma = \sigma_n, n = 1, 2, \ldots,$ the situation drastically changes, so that both $\psi(x)$ and $\psi'(x)$ become discontinuous at $x = 0$.

In order to illustrate how the isolation of the probability transmission occurs at the points $\{\sigma_n\}$ as $\epsilon \to 0$, we have plotted in figure 4 the transmitted probability $|T|^2$ as a function of σ for different small but finite values of the dimensionless parameter η. This figure clearly shows the existence of probability peaks at $\sigma = \sigma_n$, which become more and more isolated as $\eta \to 0$.

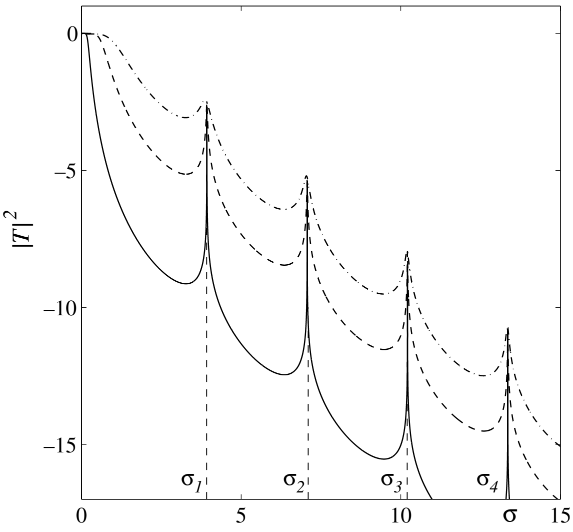

Figure 4. Logarithm of the transmission probability $|T|^2$ against σ at $\eta = 0.0005$ (solid curve), 0.05 (dashed curve) and 0.5 (dashed-dotted curve).

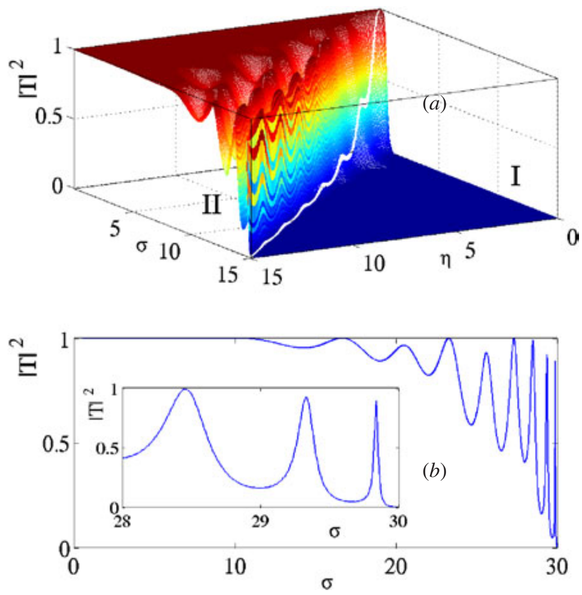

Figure 5. Transmission probability $|T|^2$ as a function of the potential strength σ and the scaling parameter η.

At the same time it is interesting to plot (see figure 5) the two-dimensional plot of the dependence of $|T|^2$ on both the parameters σ and η. As shown in figure 5, the transmission over the barrier of finite height with (dimensionless) energies η close to the barrier suffers big variation while the parameter σ changes, demonstrating a complicated wave-scattering behaviour in this region.

3. Transmission problem for the $\delta''(x)$ interaction

In the previous section, we have found that the $\delta'(x)$ potential obtained as the $\epsilon \to 0$ limit of the rectangular sequence (5) is partially transparent. Therefore, it is interesting to perform a similar limiting procedure for the $\delta''(x)$ interaction (9) using the corresponding rectangular approximation (11). In the same straightforward way one can obtain the reflection and transmission coefficients (with a finite ϵ):

$$\bar{R} = \frac{\mathrm{i}}{\bar{D}}\,\mathrm{e}^{-3\mathrm{i}\eta}\left\{\left[\left(\frac{\bar\alpha}{\bar\beta}+\frac{\bar\beta}{\bar\alpha}\right)\left(\frac{\bar\beta}{\eta}-\frac{\eta}{\bar\beta}\right)+\left(\frac{\bar\alpha}{\bar\beta}-\frac{\bar\beta}{\bar\alpha}\right)\left(\frac{\bar\beta}{\eta}+\frac{\eta}{\bar\beta}\right)\cosh(2\bar\beta)\right]\sin\bar\alpha \right.$$
$$\left. -\,2\left(\frac{\bar\beta}{\eta}+\frac{\eta}{\bar\beta}\right)\cos\bar\alpha\,\sinh(2\bar\beta)\right\} \tag{35}$$

$$\bar{T} = \frac{4}{\bar{D}}\,\mathrm{e}^{-3\mathrm{i}\eta} \qquad \eta = \epsilon k = \epsilon\sqrt{E} \tag{36}$$

where

$$\bar\alpha = \sqrt{2g/\epsilon+\eta^2} \qquad \bar\beta = \sqrt{g/\epsilon-\eta^2} \tag{37}$$

and

$$\bar{D} = 2\left[\left(\frac{\bar\beta}{\bar\alpha}-\frac{\bar\alpha}{\bar\beta}\right)\sin\bar\alpha\,\sinh(2\bar\beta)+2\cos\bar\alpha\,\cosh(2\bar\beta)\right]$$
$$+\,\mathrm{i}\left[\left(\frac{\bar\beta}{\bar\alpha}-\frac{\bar\alpha}{\bar\beta}\right)\left(\frac{\bar\beta}{\eta}-\frac{\eta}{\bar\beta}\right)\sin\bar\alpha\,\cosh(2\bar\beta)+2\left(\frac{\bar\beta}{\eta}-\frac{\eta}{\bar\beta}\right)\cos\bar\alpha\,\sinh(2\bar\beta)\right.$$
$$\left.-\left(\frac{\bar\alpha}{\bar\beta}+\frac{\bar\beta}{\bar\alpha}\right)\left(\frac{\bar\beta}{\eta}+\frac{\eta}{\bar\beta}\right)\sin\bar\alpha\right]. \tag{38}$$

It follows immediately from the expression for the reflection coefficient given by (35), (37) and (38) that $\bar{R}$ is an oscillating function with a rapidly decreasing period of oscillations as $\epsilon \to 0$. Its zeros form a *countable* set of resonance levels where the potential (10), if approximated by the rectangular family (11), is *totally* transparent. In other words, there exists an infinite sequence $\{\epsilon_n\}, n = 1, 2, \ldots,$ for which $\bar{R}(\epsilon_n) = 0$ and correspondently $\bar{T} = 1$. This sequence is given by the routes of the equation

$$\left[\left(\frac{\bar\alpha}{\bar\beta}-\frac{\bar\beta}{\bar\alpha}\right)+\left(\frac{\bar\alpha}{\bar\beta}+\frac{\bar\beta}{\bar\alpha}\right)\frac{\bar\beta^2-\eta^2}{\bar\beta^2+\eta^2}\mathrm{sech}(2\bar\beta)\right]\tan\bar\alpha = 2\tanh(2\bar\beta) \tag{39}$$

with respect to the variable ϵ, through which $\bar\alpha$, $\bar\beta$ and η are defined. For sufficiently small ϵ equation (39) is simplified to

$$\tan\sqrt{2g/\epsilon} \simeq 2\sqrt{2} \tag{40}$$

giving the solutions ϵ_n which are close to zero. However, due to this oscillating behaviour, the limit of $\bar{R}$ (or $\bar{T}$) *does not exist* as $\epsilon \to 0$. Therefore, in contrast to the case with the $\delta'(x)$ interaction (3) when the rectangular-like sequence (5) gives the well-defined limit of the scattering amplitudes, the sequence (11) that tends to the function $\delta''(x)$ does not lead to a well-defined point interaction. On the other hand, the tripole approximation by three Dirac delta functions (13) is useless as well because it leads to a point interaction with separated boundary conditions [32].

4. Summary and discussions

In this paper, we have considered both the $\delta'(x)$ and $\delta''(x)$ interaction potentials as the limits (in the sense of distributions) of appropriate families constructed from rectangles. This is the simplest choice which can easily be treated analytically yielding explicit solutions. Besides such simplicity, the use of the rectangular-like approximation can additionally be motivated as follows. First, based on physical intuition, we are able to decide how well the $\delta'(x)$ or $\delta''(x)$ distribution approximates the actual (regular) potential. Second, our procedure demonstrates how to realize explicitly the wavefunction discontinuity at $x = 0$. Third, step-like potentials can easily be manufactured using, e.g., thin layers of different types of semiconductors. More specifically, the $\delta'(x)$ interaction defined by (2) may be used as a limiting case of a device where a small region of large repulsive potential is followed by a small region of large attractive potential, which may be called a barrier–well junction. A similar experimental situation can be arranged in dielectric media.

Thus, using a particular sequence of regularizing functions constructed from rectangles in the finite dipole-like form as shown in figure 1, a countable set of resonance values for the interaction strength parameter σ at which a transparent regime of current flow occurs, has explicitly been found for the $\delta'(x)$ interaction (3). All these resonance levels, $\{\sigma_n\}_{n=1}^{\infty}$, appear to be positive roots of equation (21). This result seems to be in discrepancy with Šeba's theorem [15], according to which the transparency is identically zero and the system splits into two independent subsystems lying on the half-axes $-\infty < x < 0$ and $0 < x < \infty$. The reason for this controversy emerges from the fact that the approximating sequence (4) with $\nu = 1$ constructed from two Dirac delta functions necessary results in a continuous wavefunction with a node at $x = 0$ as $\epsilon \to 0$ blocking a current across this point. Note also that the δ' interaction in Šeba's theorem is defined by imposing the continuity of the wavefunction derivatives at the singularity. Therefore, the $\delta'(x)$ interaction studied in the present is a different object. The controversy is only apparent; there are two different self-adjoint extensions that correspond to two different sets of boundary conditions at $x = 0$ depending on which repeated limit (7) or (8) is implemented. While the repeated limit is accomplished in the way (7), one obtains a countable set $\{\sigma_n\}_{n=1}^{\infty}$ being solutions to equation (21) on the half-line $0 < \sigma < \infty$ for which a partial transmission takes place. In the second limit (8) the self-adjoint extension corresponds to the separated boundary conditions and coincides with the self-adjoint extension obtained in the limit (7) for all $\sigma \neq \sigma_n$. On the other hand, our result is embedded into the ADK [29] and TFC [30] theorems as a particular example of self-adjoint extensions corresponding to both non-separated (if $\sigma = \sigma_n$) and separated (if $\sigma \neq \sigma_n$) boundary conditions.

Another interesting feature we have observed for the rectangular approximation of the $\delta'(x)$ interaction (3) is the dependence of the reflection and transmission coefficients (17) and (18) *only* on the dimensionless scaling parameter $\eta = \epsilon k = \epsilon\sqrt{E}$ (see also (15) and (16)). As a result for each finite k these coefficients become independent of the energy $E = k^2$ in the zero-range limit $\epsilon \to 0$ because η also disappears in this limit. Similarly, the reflection and transmission coefficients for the $\delta''(x)$ interaction (10) can also be given in terms of the parameter η; however, due to equations (37) the explicit dependence on ϵ is also present. This dependence does not allow the existence of the reflection and transmission coefficients in the limit $\epsilon \to 0$. Their values oscillate between 0 and 1 as $\epsilon \to 0$, faster and faster as ϵ approaches zero.

Of course, from a mathematical point of view it is misleading to draw general conclusions based on the very particular approximations by rectangles shown in figure 1 or 3. However, from a physical point of view the rectangular approximations are the most realistic models of the point interactions or vice versa the standard distributions $\delta'(x)$ or $\delta''(x)$ as idealizations are most appropriate for finite-range potentials. On the other hand, the partial transparency of the δ'-like rectangular potential (5) with definitely clear peaks shown in figure 4 at some potential strength values deserves attention, for instance, from the point of view of technological applications.

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